Double sweep method for tri-diagonal system of linear equations (Thompson's method)

Problem formulation

The solution of a system of linear algebraic equations is sought

(1) $Ax = d$,

where A is a triple matrix, x – vector of the unknown quantities, d – right side:

(2) $$A = \begin{pmatrix} b_1 & c_1 & 0 & \dots & & & 0 \\ a_2 & b_2 & c_2 & 0 & \dots & & 0 \\ & & \dots & & & & \\ 0 & \dots & a_k & b_k & c_k & \dots & 0 \\ & & & & \dots & & \\ 0 & \dots & & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \dots & & & o & a_n & b_n \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \\ x_k \\ \\ \dots \\ x_n \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \\ d_k \\ \\ \dots \\ d_n \end{pmatrix} \neq 0 .$$

In an extended form the system is:

(3) $$\left| \begin{array}{llllllll} b_1 x_1 & + c_1 x_2 & & & & & = & d_1 \\ a_2 x_1 & + b_2 x_2 & + c_2 x_3 & & & & = & d_2 \\ & a_3 x_2 & + b_3 x_3 & + c_3 x_4 & & & = & d_3 \\ & & & \dots\dots\dots & & & & \dots\dots \\ & & & a_k x_{k-1} & + b_k x_k & + c_k x_{k+1} & = & d_k \\ & & & & \dots\dots\dots & & & \dots\dots \\ & & & & a_{n-1} x_{n-2} & + b_{n-1} x_{n-1} & + c_{n-1} x_n = & d_{n-1} \\ & & & & & a_n x_{n-1} & + b_n x_n = & d_n \end{array} \right. .$$

This system can be solved using any of the existing methods for solving systems of linear algebraic equations: method of Gauss, the Gauss-Jordan method, simple iteration method and so on. But since it is of a special kind for its solution special and effective methods are created to solve it, among which the double sweep method. The method belongs to the group of exact methods, i.e. theoretically using exact numbers (without rounding off) after a finite number of arithmetical operations the result is an exact solution. However, in the case of real systems, as a rule the number of equations n is too great and inevitably rounding off up to a certain digit is used for the intermediate results. This can lead to error accumulation, regardless of the method being exact. In this sense the following theorem is of importance:

A sufficient condition for the stability of the double sweep method

Let

(4) $|b_k| \geq |a_k| + |c_k|, \quad k = \overline{1,n}.$

Then there exists a single solution to system (1) which is also stable, i.e. insignificant errors in the initial data lead to insignificant errors in the result.

Note 1. Condition (4) can be made more precise. It is not hard to realize that in fact it signifies a dominating principle diagonal of matrix *A*.

Note 2. It is possible to have cases when the double sweep method works without the stability condition being fulfilled, in other words it is sufficient but not necessary.

Calculation scheme for the double sweep method

For the sake of convenience, two auxiliary zero unknowns $x_0 = 0,\ x_{n+1} = 0$ are introduced which help to complement the first and the last equation of system (3), when coefficients a_1 and c_n are random, so that all rows have the same form.

We substitute:

(5) $x_k = \alpha_k x_{k+1} + \beta_k \ , \quad k = \overline{1,n-1},$

where α_k, β_k are the double sweep coefficients. They are consecutively calculated using the following recurrent formulas:

(6) $$\boxed{\begin{array}{l} \alpha_1 = -\dfrac{c_1}{b_1}, \ \ \beta_1 = \dfrac{d_1}{b_1}, \\ \alpha_k = -\dfrac{c_k}{b_k + a_k\alpha_{k-1}}, \quad \beta_k = \dfrac{d_k - a_k\beta_{k-1}}{b_k + a_k\alpha_{k-1}}, \quad k = \overline{2,n} \end{array}}$$

After determining them, implementing (5) in reverse order, we find the sought solutions of the system:

(7) $$\boxed{x_n = \beta_n\,, \quad x_k = \alpha_k x_{k+1} + \beta_k\,, \quad k = n-1, n-2, \ldots, 1\,.}$$

Example. Using the double sweep method solve the following system

$$\left|\begin{array}{lllllr} 2x_1 & +x_2 & & & = & 8 \\ -x_1 & +2x_2 & -x_3 & & = & 3{,}2 \\ & 2x_2 & -4x_3 & & = & -0{,}5 \\ & & -2x_3 & +4x_4 & = & 2 \end{array}\right. .$$

Solution:

It is done in three stages:

$\boxed{1.}$ Check for the stability condition (4),

$\boxed{2.}$ Calculation of double sweep coefficients using formulas (6),

$\boxed{3.}$ Calculation of the unknowns in a reverse order using formulas (7).

We begin with **stage** $\boxed{\mathbf{1.}}$ We have $n = 4$. We compile a table for the coefficients:

(8)

k	a_k	b_k	c_k	d_k
1	0	2	1	8
2	-1	2	-1	3,2
3	2	-4	0	-0,5
4	-2	4	0	2

It is easy to check conditions (4) respectively for $k = 1,2,3,4$: $|b_1| \geq |a_1| + |c_1|$ since $|2| \geq |0| + |1|$; $|b_2| \geq |a_2| + |c_2|$ since $|2| \geq |-1| + |-1|$; $|b_3| \geq |a_3| + |c_3|$ from $|-4| \geq |2| + |0|$ and $|b_4| \geq |a_4| + |c_4|$, because $|4| \geq |-2| + |0|$. Consequently the condition for stability has been satisfied.

Stage $\boxed{\mathbf{2.}}$ We calculate the double sweep coefficients using formulas (6) and notice that the denominators $z_k = b_k + a_k \alpha_{k-1}$ for a fixed *k* are the same:

$\alpha_0 = 0,\ \beta_0 = 0\,,$

$z_1 = b_1 + a_1 \alpha_0 = 2 + (-1).0 = 2\,,$

$$\alpha_1 = -\frac{c_1}{b_1 + a_1 \alpha_0} = -\frac{1}{z_1} = -\frac{1}{2} = -0{,}5\ , \qquad \beta_1 = \frac{d_1 - a_1 \beta_0}{z_1} = \frac{8 - 0.0}{2} = 4\,;$$

$z_2 = b_2 + a_2 \alpha_1 = 2 + (-1).(-0{,}5) = 2{,}5\ ,$

$$\alpha_2 = -\frac{c_2}{b_2 + a_2\alpha_1} = -\frac{-1}{z_2} = -\frac{1}{2{,}5} = 0{,}4, \qquad \beta_2 = \frac{d_2 - a_2\beta_1}{b_2 + a_2\alpha_1} = \frac{3{,}2 - (-1).4}{2{,}5} = \frac{7{,}2}{2{,}5} = 2{,}88\ ;$$

$$z_3 = b_3 + a_3\alpha_2 = -4 + 2.(0{,}4) = -3{,}2\,,$$

$$\alpha_3 = -\frac{c_3}{b_3 + a_3\alpha_2} = -\frac{0}{z_3} = 0\,, \qquad \beta_3 = \frac{d_3 - a_3\beta_2}{b_3 + a_3\alpha_2} = \frac{-0{,}5 - 2.(2{,}88)}{-3{,}2} = \frac{-6{,}26}{-3{,}2} = 1{,}95625\,;$$

$$z_4 = b_4 + a_4\alpha_3 = 4 + (-2).0 = 4\,,$$

$$\alpha_4 = -\frac{c_4}{b_4 + a_4\alpha_3} = -\frac{0}{z_3} = 0\,,\ \beta_4 = \frac{d_4 - a_4\beta_3}{b_4 + a_4\alpha_3} = \frac{2 - (-2).(1{,}95625)}{4} = \frac{5{,}9125}{4} = 1{,}478125\,.$$

It is convenient to enter all results into a table as shown below.

Table 1

k	a_k	b_k	c_k	d_k	z_k	α_k	β_k
1	0	2	1	8	2	-0,5	4
2	-1	2	-1	3,2	2,5	0,4	2,88
3	2	-4	0	-0,5	-3,2	0	1,95625
4	-2	4	0	2	4	0	1,478125

Stage $\boxed{3.}$ We calculate the unknowns using formulas (7):

$x_4 = \beta_4 = 1{,}478125\ ;$

$x_3 = \alpha_3 x_4 + \beta_3 = 0.(1{,}478125) + 1{,}95625 = 1{,}95625\ ;$

$x_2 = \alpha_2 x_3 + \beta_2 = (0{,}4).(1{,}95625) + 2{,}88 = 3{,}6625\ ;$

$x_1 = \alpha_1 x_2 + \beta_1 = (-0{,}5).(3{,}6625) + 4 = 2{,}16875\,.$

Answer: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2{,}16875 \\ 3{,}6625 \\ 1{,}95625 \\ 1{,}478125 \end{pmatrix}.$

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